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RELATIONS

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- Cartesian product – Cartesian product of 2 non-empty sets A & B is defined as

$$A \times B = \{(a,b) : a \in A,\ b \in B\}$$

eg – $A = \{a, b, c\}$
$B = \{1, 2, 3\}$

$\Rightarrow A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$

NOTE: ① If either A or B is null set, then $A \times B = \emptyset$

② If $\#elem(A) = p$ & $\#elem(B) = q$,
$\Rightarrow \#elem(A \times B) = pq$

- Reln – A reln R from a non-empty set A to a non-empty set B is subset of cartesian product $A \times B$.

(R is a reln from A to B) $\Leftrightarrow$ $\boxed{R \subseteq A \times B}$

If #elem(A) = m & #elem(B) = n, then total #relⁿ from A to B is 2^{mn}

- **Domain & Range of Relⁿ** – $D(R) = \{a : (a,b) \in R\}$
  $R(R) = \{b : (a,b) \in R\}$

eg – $R = \{(a,1), (b,2), (b,3), (c,1)\}$
$D(R) = \{a, b, c\}$
$R(R) = \{1, 2, 3\}$

TYPES OF RELⁿ

① **Reflexive** – A relⁿ R on a set A is said to be reflexive if every elem. of A is related to itself.

eg – $A = \{1, 2, 3\}$
$R_1 = \{(1,1), (1,2), (2,2), (3,3), (1,3)\}$
$R_2 = \{(1,1), (1,3), (2,2)\}$

$R_1 \rightarrow$ Reflexive $\qquad R_2 \rightarrow$ NOT reflexive

Thus, R on the set A is not reflexive if $\exists\, a \in A$ s.t. $a \not R a$

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(2) Symmetric — A relⁿ R on a set A is said to be symmetric iff,

$$aRb \Rightarrow bRa \quad \forall \; a,b \in A$$

eg — $A = \{1,2,3,4\}$

$R_1 = \{(1,3),(1,4),(2,2),(4,1),(3,1)\}$

$R_2 = \{(1,1),(2,2),(3,3),(1,3)\}$

$R_1 \to$ Symmetric $\quad$ $R_2 \to$ NOT symmetric

(3) Transitive — A relⁿ R on set A is said to be a transitive relⁿ iff.

$$(aRb) \wedge (bRc) \Rightarrow aRc \;, \quad \forall \; a,b,c \in A$$

eg — $xRy \Leftrightarrow x < y \;, \quad \forall \; x,y \in \mathbb{N}$

$R \to$ Transitive

(4) Equivalence relⁿ — A relⁿ R on a set A is said to be an eq. relⁿ iff it is reflexive, symmetric & transitive.

(5) <u>Identity rel^n</u> — A rel^n R on A s.t.

$$R = \{(a,a) : a \in A\}, \quad \forall a \in A$$

<u>Case-I</u> : If A is non-empty set, then a rel^n R on set A where $R = \emptyset$

$R \longrightarrow$ Symmetric, Transitive but <u>not</u> Reflexive

<u>Case-II</u> : If A is empty set, the a rel^n R on set A where $R = \emptyset$

$R \longrightarrow$ Reflexive, Symmetric & Transitive
$\Rightarrow$ <u>Eq. rel^n</u>

Q. Prove that a rel^n R on $\mathbb{Z}$,

$x R y \Leftrightarrow (x-y)$ is divisible by n

is an eq. rel^n.

<u>A.</u> (I) $x - x = 0$; $\because n \mid 0 \Rightarrow x R x$

(II) let $xRy \Rightarrow n \mid x-y \Rightarrow n \mid (y-x) \Rightarrow yRx$

(III) let xRy & yRz $\Rightarrow n \mid (x-y)$ & $n \mid (y-z)$ $\Rightarrow n \mid (x-y+y-z) \Rightarrow n \mid (x-z)$
$\Rightarrow xRz$

Hence, eq. rel^n

FUNCTIONS

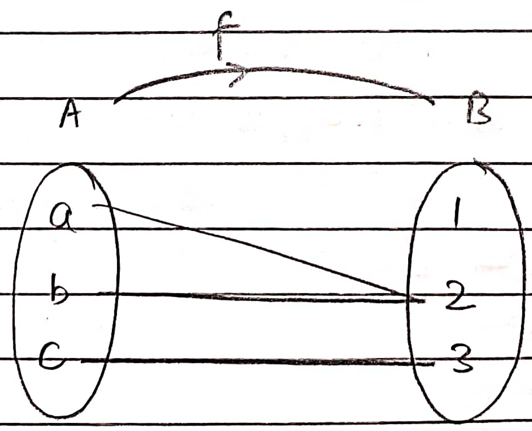
27/04/2023

Def^n - Consider 2 non-empty sets A & B.
A function $f: A \rightarrow B$ is a rule or correspondence which associates each elem. of set A. to a unique elem. of set B.

We write this correspondence as $f: A \rightarrow B$.
Thus a fn^n f from a set A to a set B is a subset to $A \times B$ in which each $a \in A$ appears in one and only one ordered pair of f.

- If an elem. $a \in A$ is associated with an elem. $b \in B$, then b is called the 'f-image of a' or 'image of a under rule f' or the value of fn 'f' at a.

Also, a is called the pre-image of b under the rule f



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Here, set A is known as domain of the f^n f.

Domain is the set of pts. where f^n is supposed to be well-defined.

Here, set B is known as co-domain.

Range is the set of pts. $\in B$ which have pre-image in set A.

$$\boxed{\text{Range} \subseteq \text{Co-domain}}$$

EXAMPLES OF FX^ns

(1) Polynomial f^n –

$$f(x) = a_n x^n + a_{(n-1)} x^{(n-1)} + \ldots + a_0 \quad (a_n \neq 0)$$

Domain = $\mathbb{R}$
Range :

n	$a_n > 0$	$a_n < 0$
Odd	$\mathbb{R}$	$\mathbb{R}$
Even	Bounded below	Bounded above

(2) Algebraic f^n — $f(x) = \sqrt{x^2+1}$, $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x^2}$

(3) Rational f^n — $f(x) = \frac{P(x)}{Q(x)}$

(4) Identity f^n — $f(x) = x$

(5) Const f^n — $f(x) = c$; $c \in \mathbb{R}$.

Domain — $\mathbb{R}$ Range — $\{c\}$

(6) Trigonometric f^n —

f^n	Domain	Range
$\sin(x)$	$\mathbb{R}$	$[-1, 1]$
$\cos(x)$	$\mathbb{R}$	$[-1, 1]$
$\tan(x)$	$\mathbb{R} - \left\{\frac{(2n-1)\pi}{2}\right\}$	$(-\infty, \infty)$
$\cot(x)$	$\mathbb{R} - \{n\pi\}$	$(-\infty, \infty)$
$\sec(x)$	$\mathbb{R} - \left\{\frac{(2n-1)}{2}\pi\right\}$	$(-\infty, -1] \cup [1, \infty)$
$\operatorname{cosec}(x)$	$\mathbb{R} - \{n\pi\}$	$(-\infty, -1] \cup [1, \infty)$

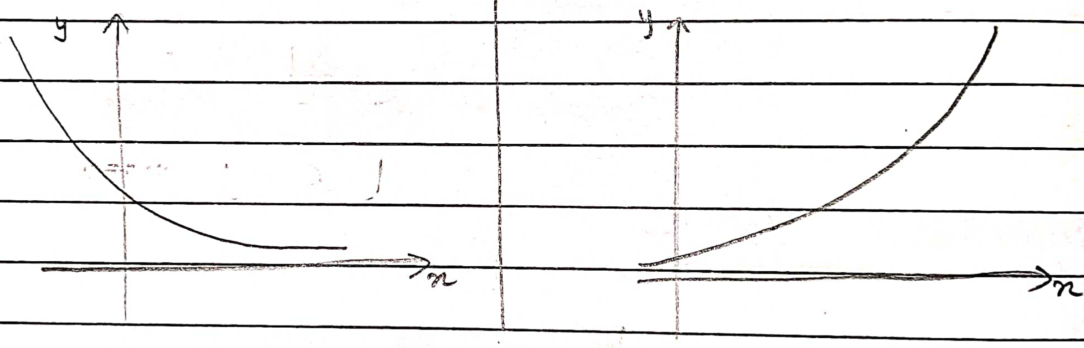
Fxn	Domain	Range
$\sin^{-1}(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$\cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \{\frac{\pi}{2}\}$
$\operatorname{cosec}^{-1}(x)$	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$

(7) Exponential fxn – $f: \mathbb{R} \rightarrow \mathbb{R}$

$f(x) = a^x$; $a \rightarrow$ const.

$a > 0$ & $a \neq 1$

$0 < a < 1$ | $a > 1$



Range – $(0, \infty)$

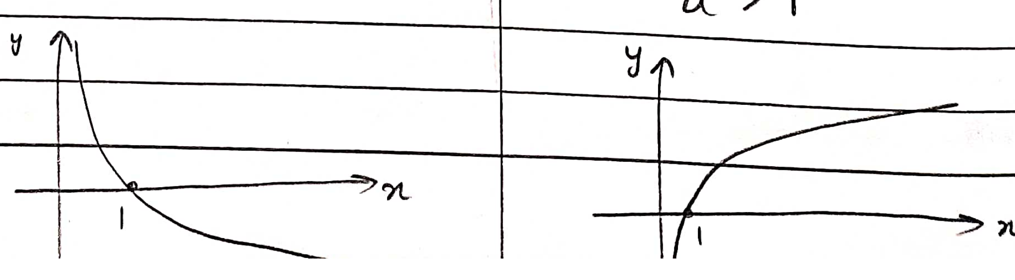
(8) Logarithmic fxn – $f: \mathbb{R}^+ \longrightarrow \mathbb{R}$

Range – $\mathbb{R}$

$f(x) = \log_a(x)$; $a \rightarrow$ const

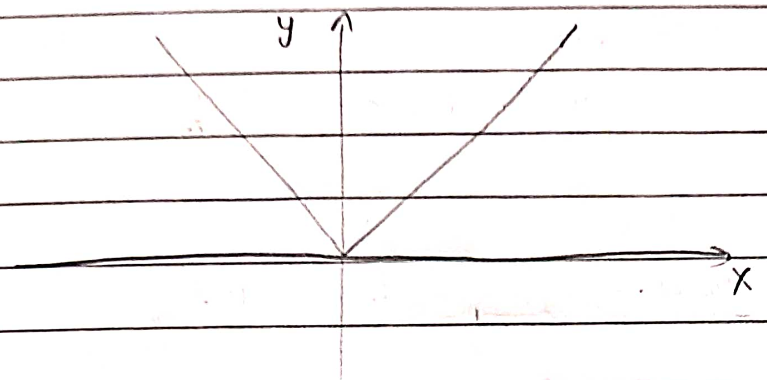
$a > 0$ & $a \neq 1$

$0 < a < 1$ | $a > 1$



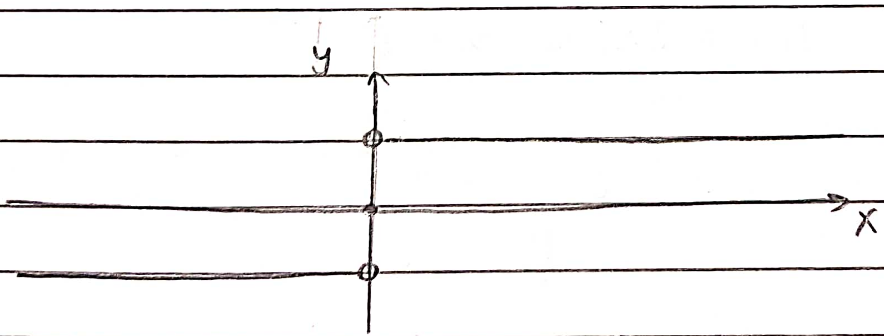
(9) Modulus $f x^n$ — $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = |x|$$



Range — $[0, \infty)$

(10) Signum $f x^n$ — $sgn(x) = \begin{cases} \frac{|x|}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$



Range — $\{-1, 0, 1\}$

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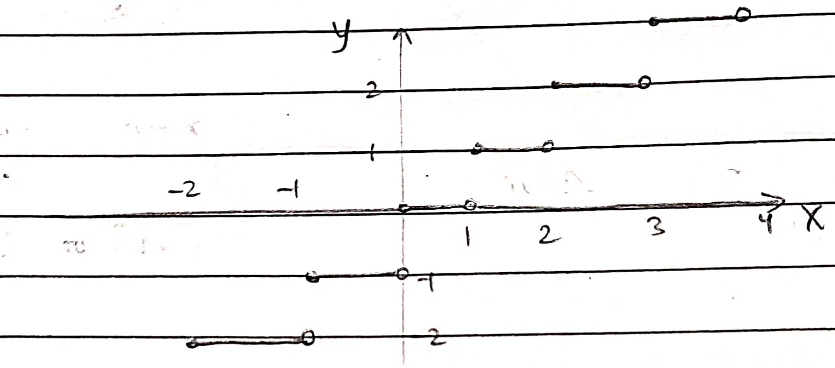
(11) Greatest Integer $f x^n$ (GIF) –

$f : \mathbb{R} \to \mathbb{R}$ $f(x) = [x]$

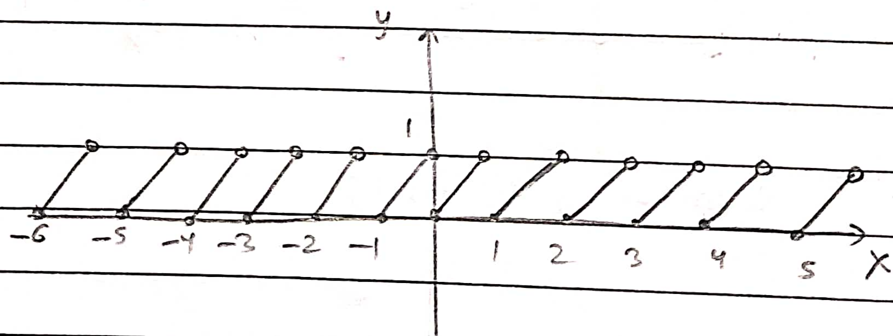
greatest Integer of $x \le x$



Range – $\mathbb{Z}$

(12) Fractional Part $f x^n$ – $f : \mathbb{R} \to \mathbb{R}$

$f(x) = \{x\} = x - [x]$



Range – $[0, 1)$

→ Ppts of GIF & Fractional Part fn^n

(1) $[[x]] = [x]$

(2) $0 \leq [x] < 1$

(3) $[\{x\}] = 0$

$\{[x]\} = 0$

(4) $x-1 < [x] \leq x$

$[x] \leq x < [x]+1$

(6) $[x \pm n] = [x] \pm n$, $n \in \mathbb{Z}$

$[nx] \neq n[x]$

(5) $[x] + [-x] = \begin{cases} 0 , & x \in \mathbb{Z} \\ -1 , & x \notin \mathbb{Z} \end{cases}$

$[-x] = \begin{cases} -x , & x \in \mathbb{Z} \\ -1-[x] , & x \notin \mathbb{Z} \end{cases}$

(7) $\{x\} + \{-x\} = \begin{cases} 0 , & x \in \mathbb{Z} \\ 1 , & x \notin \mathbb{Z} \end{cases}$

(8) $[x]+[y] \leq [x+y] \leq [x]+[y]+1$

☆ (9) $[x] + \left[x+\frac{1}{n}\right] + \left[x+\frac{2}{n}\right] + \dots + \left[x+\frac{n-1}{n}\right] = [nx]$; $n \in \mathbb{N}$

(10) $\{-x\} = 1 - \{x\}$, $x \notin \mathbb{Z}$

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Q. ① Given $y = 2[x] + 3$ & $y = 3[x-2] + 5$, find $[x+y]$

② Solve (i) $4[x] = x + \{x\}$

(ii) $x^2 - 4x + [x] + 3 = 0$

(iii) $x^2 - 4 - [x] = 0$

(iv) $|x^2 - 1 + \sin x| = |x^2 - 1| + |\sin x|$; $x \in [-2\pi, 2\pi]$

③ If $n\{x\} = x + [x]$; $n \in \mathbb{N}$, $n > 1$, has exactly 5 solⁿs, then $n = ?$

A. ① $2[x] + 3 = 3[x] - 6 + 5 \Rightarrow [x] = 4$

$y = 11 \Rightarrow [x+y] = [x] + y = 15$

② (i) $4(x - \{x\}) = x + \{x\} \Rightarrow \underbrace{\{x\}}_{\in [0,1)} = \left(\frac{3}{5}\right)x \Rightarrow x \in [0, \frac{5}{3}]$

a. $x \in [0,1) \Rightarrow x = \frac{3}{5}x \Rightarrow \boxed{x = 0}$

b. $x \in [1, \frac{5}{3}) \Rightarrow x - 1 = \frac{3x}{5} \Rightarrow \boxed{x = \frac{5}{2}}$ X

(ii) $x^2 - 4x + x - \{x\} + 3 = 0 \Rightarrow x^2 - 3x + 3 = \{x\}$

$\Rightarrow \left(x - \frac{3}{2}\right)^2 + \frac{3}{4} = \{x\}$

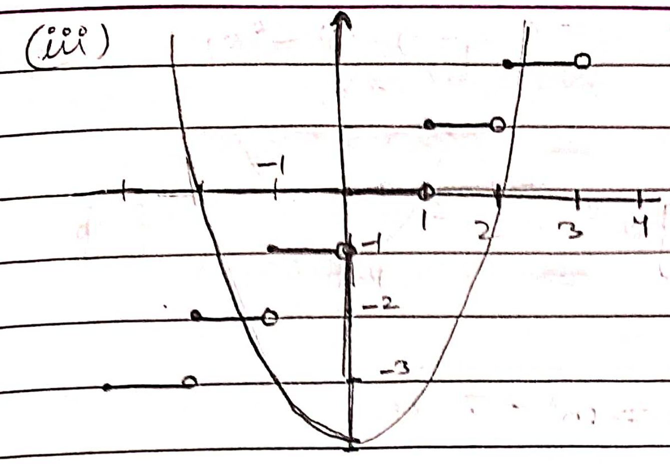
$\{x\} \in [0,1) \Rightarrow \left(x - \frac{3}{2}\right)^2 \in [0, \frac{1}{4}) \Rightarrow \left(x - \frac{3}{2}\right) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

$\Rightarrow \boxed{x \in (1, 2)}$

$\Rightarrow x^2 - 3x + 3 = (x - 1)$

$\Rightarrow x^2 - 4x + 4 = 0 \Rightarrow \boxed{x = \pm 2}$ X $\Rightarrow$ No solⁿ.

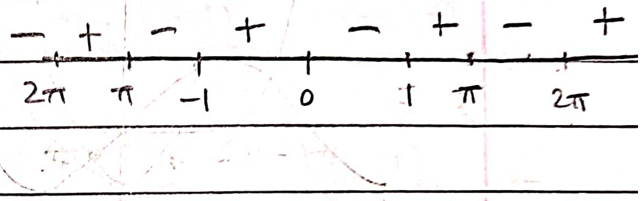
(iii)



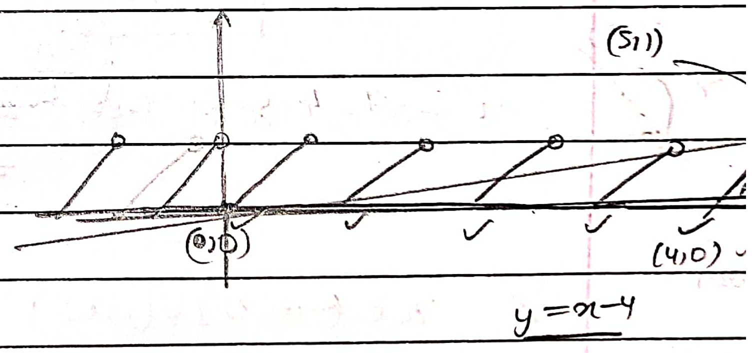
a. $x^2 - 4 = 2 \Rightarrow \boxed{x = \sqrt{6}}$

b. $x^2 - 4 = -2 \Rightarrow \boxed{x = -\sqrt{2}}$

(iv) $\underbrace{(x^2-1)\,(\sin x)}\geq 0$



5) $n\{x\} = x + (x - \{x\}) \Rightarrow \{x\} = \left(\frac{2}{n+1}\right)x$



a. $m > \left(\frac{1}{6}\right)$

b. $m < \left(\frac{1}{5}\right)$

$\Rightarrow \frac{1}{6} < \frac{2}{(n+1)} < \frac{1}{5}$

$\Rightarrow 6 > \left(\frac{n+1}{2}\right)$ $\qquad \left(\frac{n+1}{2}\right) > 5$

$\Rightarrow n < 12 \qquad \Rightarrow n > 9 \qquad \Rightarrow \boxed{n = 10, 11}$

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NOTE: Increasing fxn : $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Decreasing fxn : $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

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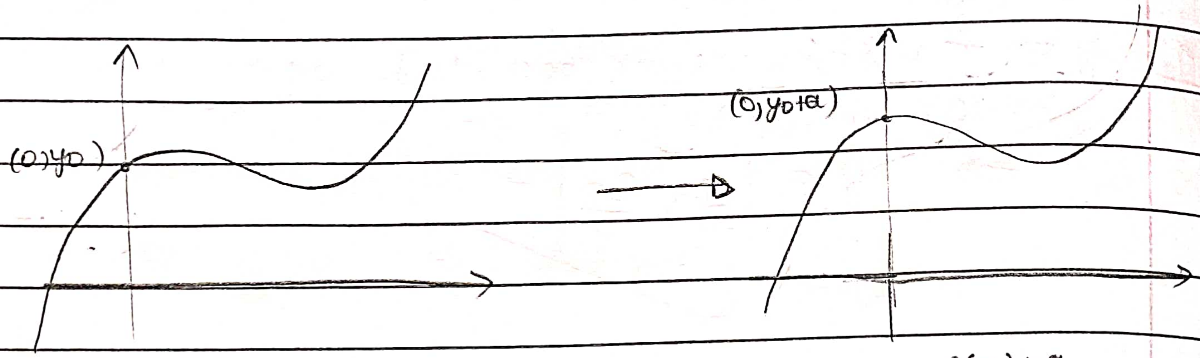
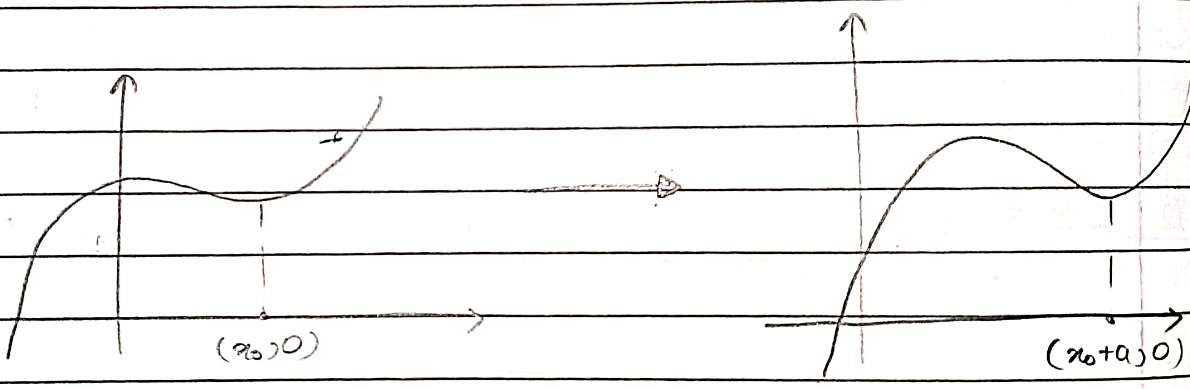
GRAPHICAL TRANSFORMATIONS

① $f(x) \pm a \quad ; \quad a > 0$



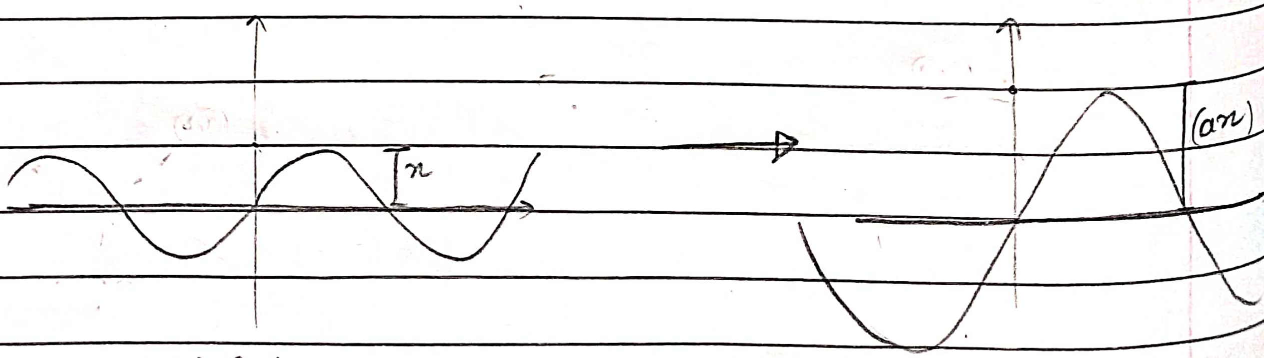
$f(x)$ $\qquad$ $f(x)+a$

② $f(x \pm a)$



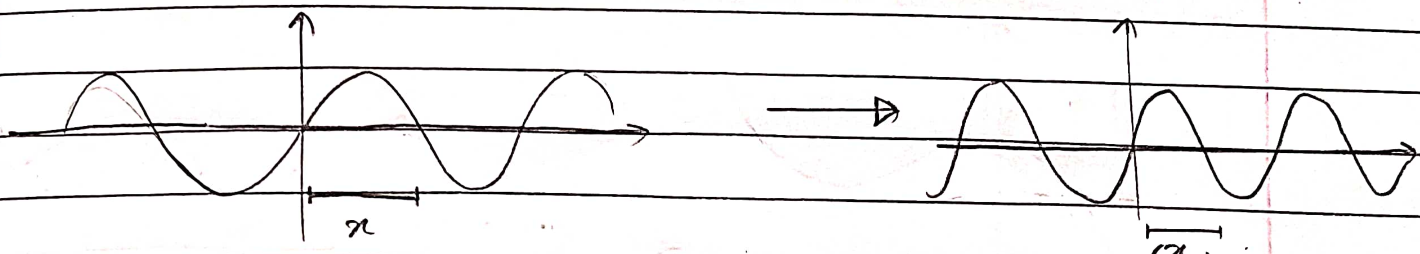
$f(x)$ $\qquad$ $f(x-a)$

③ $a \cdot f(x) \quad ; \quad a > 0$



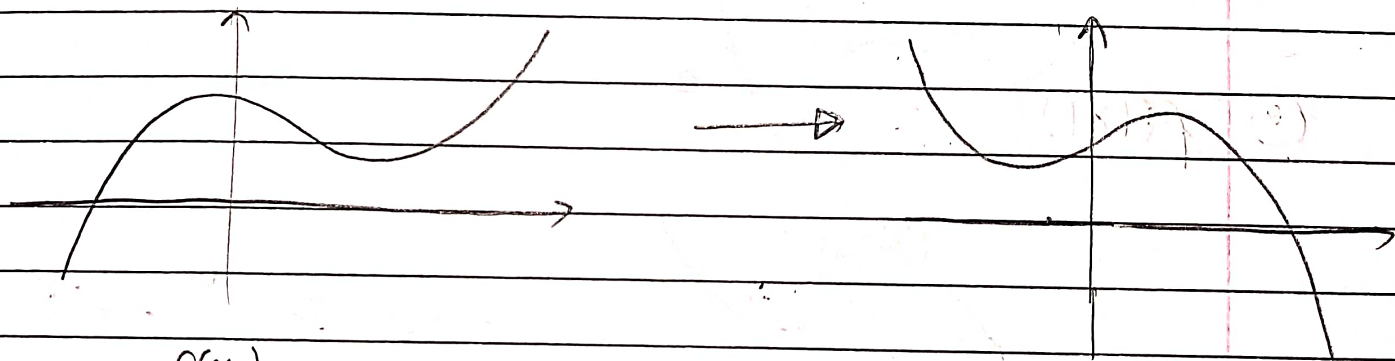
$\sin(x)$ $\qquad$ $a\sin(x)$

4. $f(ax)$; $a > 0$



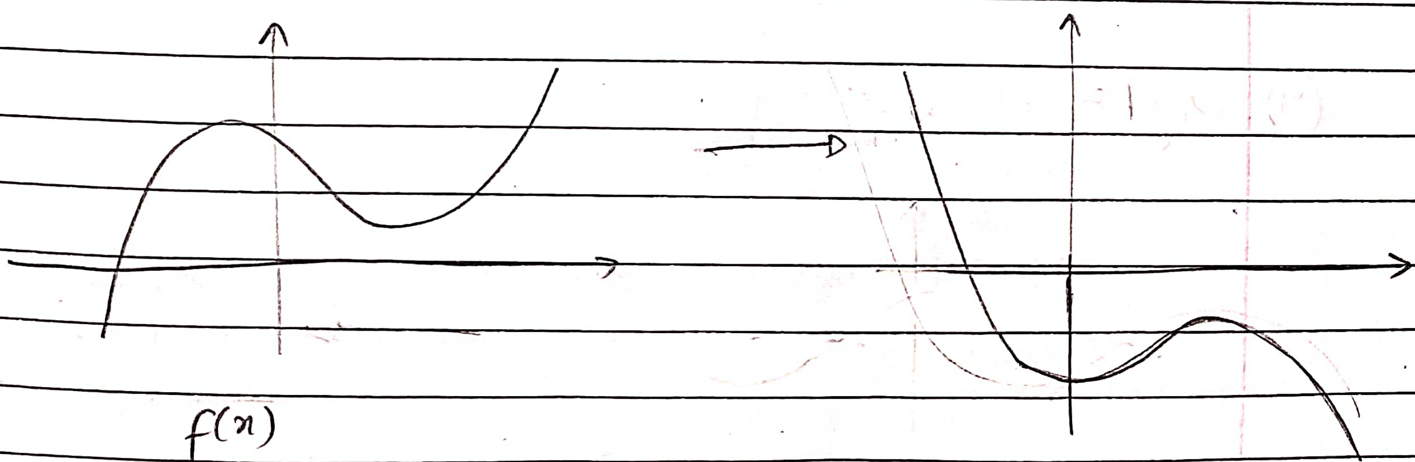
$\sin(x)$ $\quad\quad$ $\sin(ax)$

5. $f(-x)$



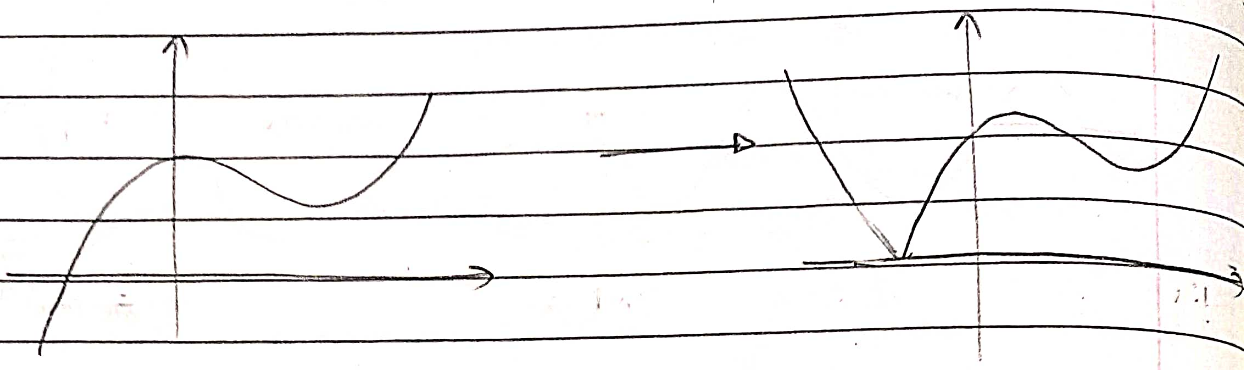
$f(x)$ $\quad\quad$ $f(-x)$

6. $-f(x)$



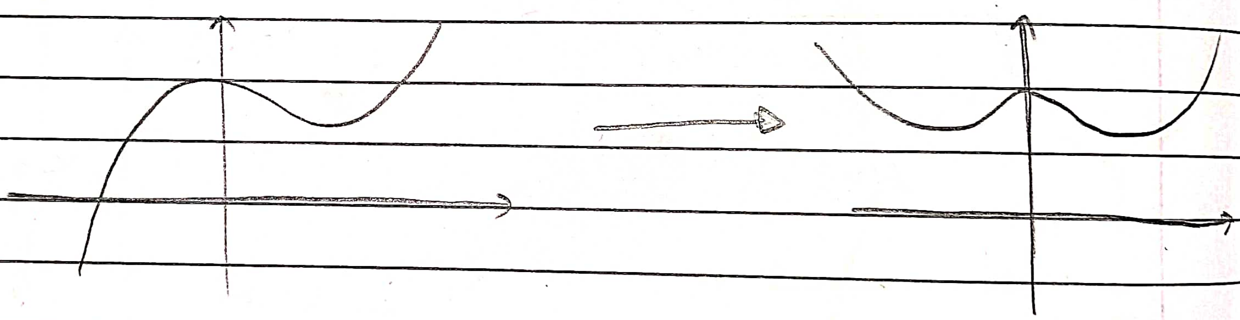
$f(x)$ $\quad\quad$ $-f(x)$

(7) $|f(x)|$



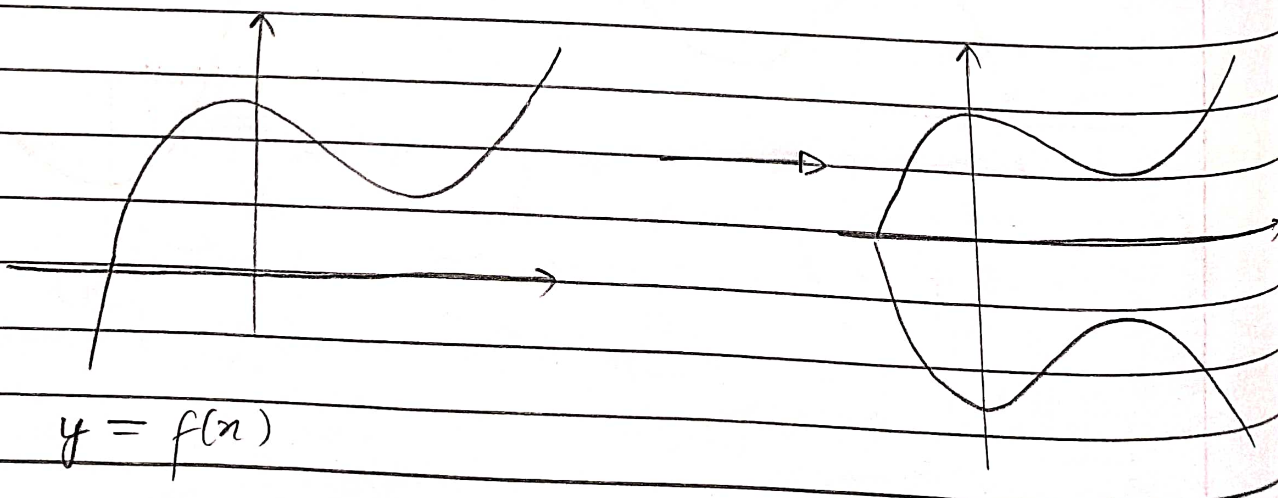
$f(x)$ $\quad$ $|f(x)|$

(8) $f(|x|)$



$f(x)$ $\quad$ $f(|x|)$

(9) $|y| = f(x)$

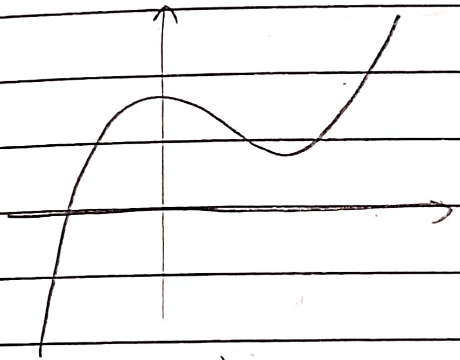


$y = f(x)$ $\quad$ $|y| = f(x)$

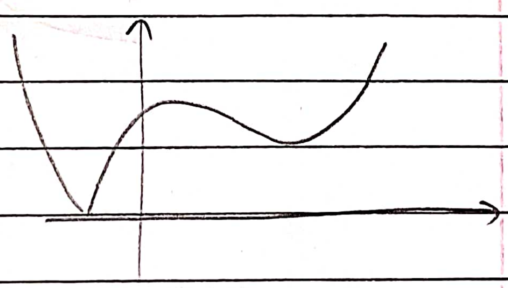
(10) $|y| = |f(x)|$

$y = f(x) \longrightarrow y = |f(x)| \longrightarrow |y| = |f(x)|$

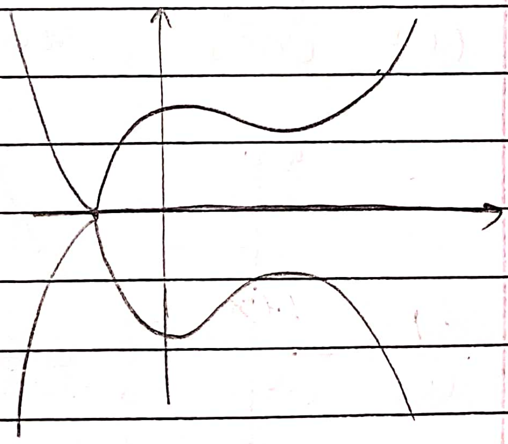
<u>NOT</u> $y = f(x) \longrightarrow |y| = f(x) \longrightarrow |y| = |f(x)|$



$f(x)$

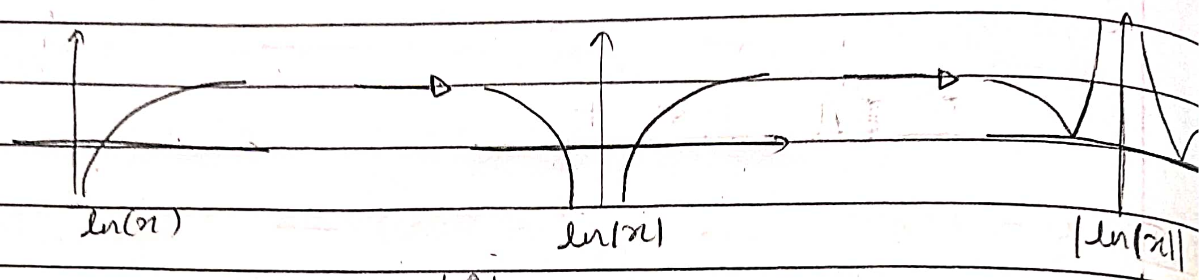


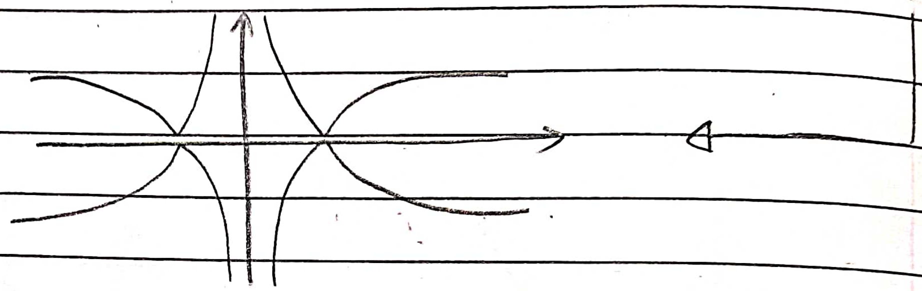
$|f(x)|$



$|y| = |f(x)|$

eg. $|y| = |\ln|x||$

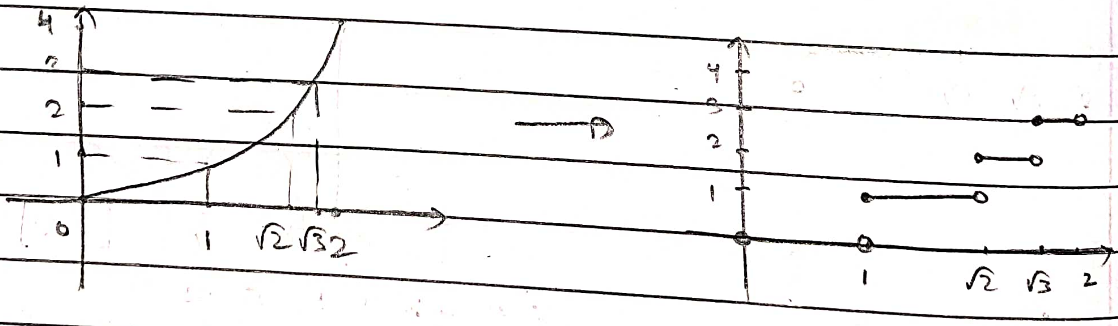




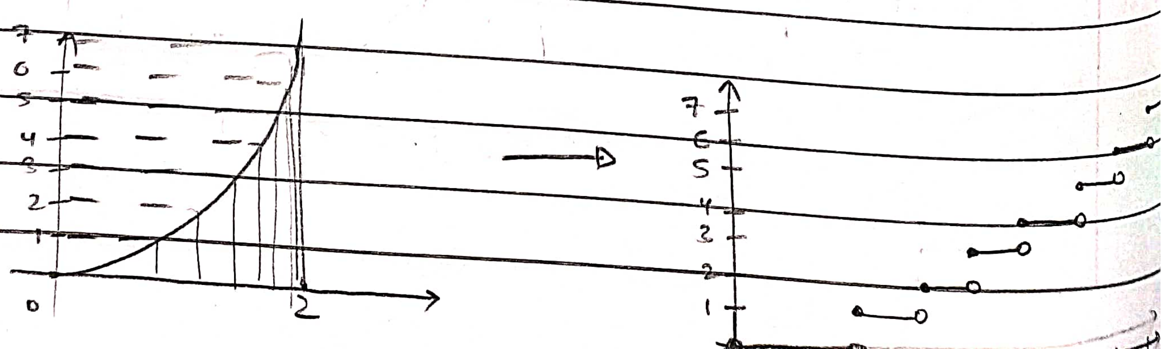
$|y| = |\ln|x||$

(11) $[f(x)]$

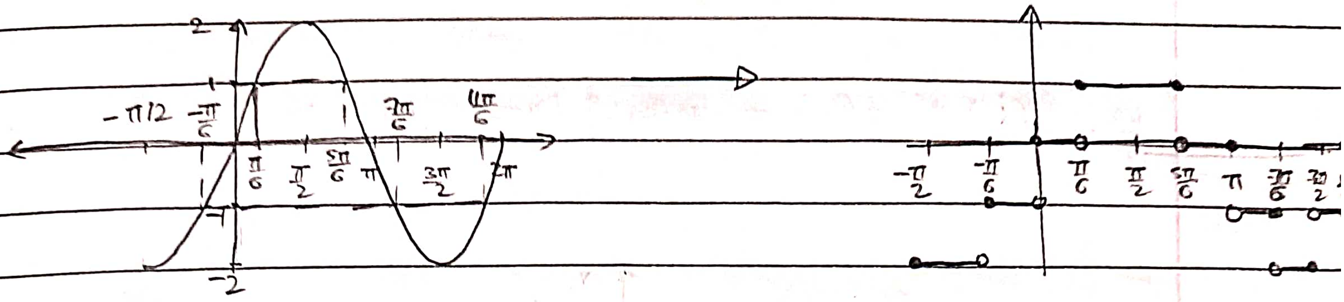
(i) $[x^2]$, $x \in [0, 2)$



(ii) $[x^3]$, $x \in [0, 2)$

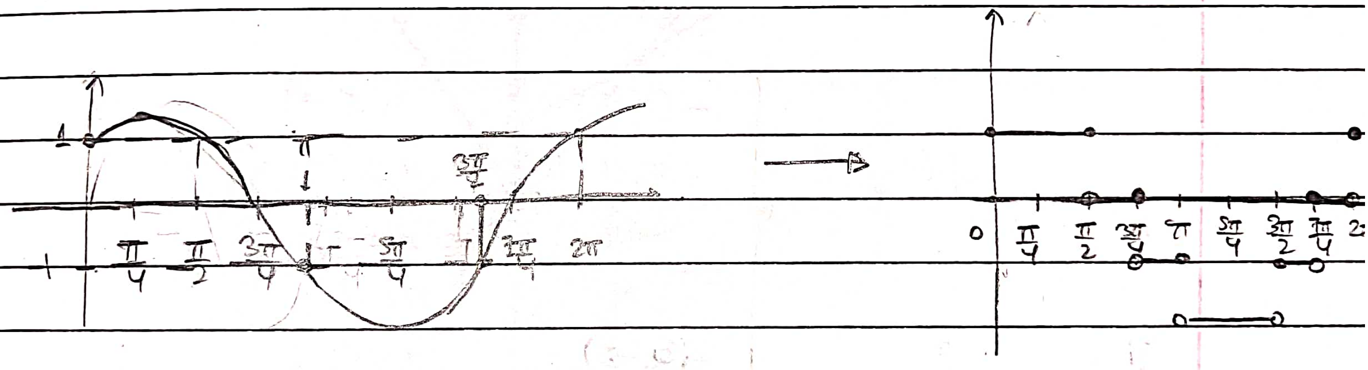


(iii) $[2\sin x]$; $x \in \left[-\frac{\pi}{2}, 2\pi\right]$



(iv) $[\sin x + \cos x]$; $x \in [0, 2\pi]$.

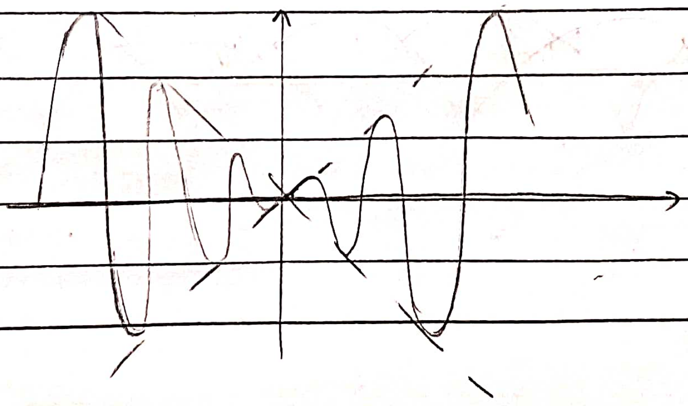
$\Rightarrow \left[\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)\right]$



(12) $f(x)\sin x$

eg: $x\sin x$ $\qquad -1 \le \sin x \le 1$

☆ $\Rightarrow -x \le \sin x \le x$



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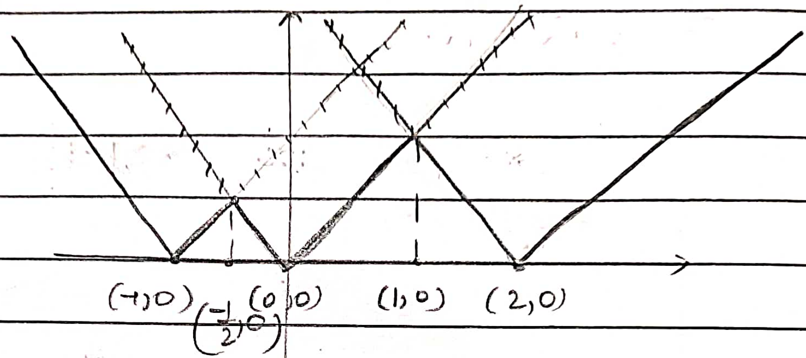
(13) $h_1(x) = \max\{f(x), g(x)\}$

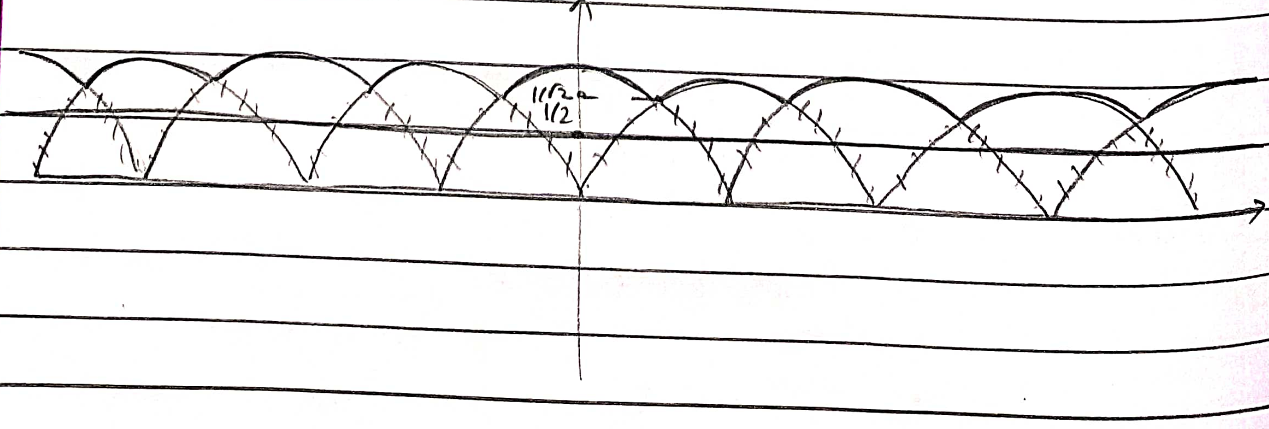
$h_2(x) = \min\{f(x), g(x)\}$

eg (i) $f(x) = \min.\{|x-2|, |x|, |x+1|\}$



$$f(x) = \begin{cases} -(x+1) & ; \quad x \le -1 \\ x+1 & ; \quad x \in (-1, -\frac{1}{2}] \\ -x & ; \quad x \in (-\frac{1}{2}, 0] \\ x & ; \quad x \in (0, 1] \\ -(x-2) & ; \quad x \in (1, 2] \\ x-2 & ; \quad x \in (2, \infty) \end{cases}$$

(ii) $f(x) = \max\{|\sin x|, |\cos x|, \frac{1}{2}\}$



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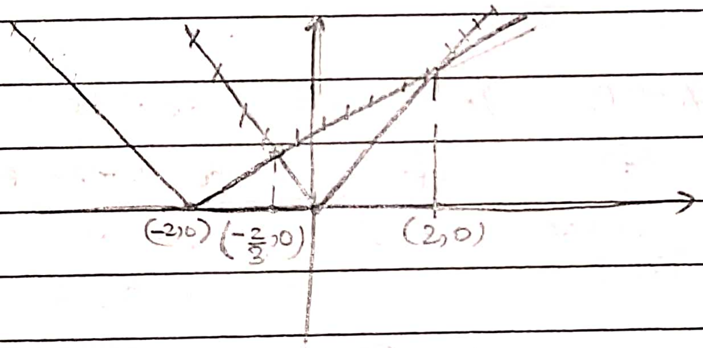
NOTE: (1) $\max\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} + \left|\frac{f(x)-g(x)}{2}\right|$

(2) $\min\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} - \left|\frac{f(x)-g(x)}{2}\right|$

Q. Draw the graph of

$$f(x) = 2|x| + |x+2| - \big||x+2| - 2|x|\big|$$
$$= \min\{4|x|, 2|x+2|\}$$



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TYPES OF FUNCTIONS

(1) One – one (Injective) & Many – one fn –

A fn $f : A \to B$ is said to be a one-one fn iff diff. elem. of A have diff. f-images in B

Thus, $\forall$ $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$.

$$f(x_1) = f(x_2) \iff x_1 = x_2$$

OR

$$f(x_1) \neq f(x_2) \iff x_1 \neq x_2$$

A fn $f : A \to B$ is said to be a many-one fn if 2 or more elem. of A have the same f-image in B.

Thus, $\exists$ $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$

$$[f(x_1) = f(x_2)] \wedge [x_1 \neq x_2]$$

(2) Into & Onto (Surjective) fn –

If a $f : A \to B$ is s.t each elem in B (co-domain) is the f-image of at least one elem. in A, then the f is said to be a fn of A onto B.

Thus, $\forall\ b \in B,\ \exists\ a \in A\ \ [f(a) = b]$

NOTE: If $\boxed{\text{Range} = \text{Co-domain}} \iff f$ is an <u>onto f^n</u>.

If a $f: A \to B$ is s.t $\exists$ at least one elem in B (co-domain), which is <u>not</u> the image of any elem in A (domain), then f is said to be an <u>into f^n</u>.

Thus, $\exists\ b \in B,\ \forall\ a \in A\ \ [f(a) \neq b]$

	<u>One-one</u>	<u>Many-one</u>
<u>Onto</u>	Injective & Surjective (Bijective)	Surjective but <u>not</u> Injective
<u>Into.</u>	Injective but <u>not</u> Surjective	Neither Injective nor Surjective

<u>eg</u>:- $f: A \to B$; $f(x) = \sin x$.
Define A & B s.t f is.

(i) $I\ \&\ \neg S \longrightarrow A = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $B = \mathbb{R}$
(ii) $\neg I\ \&\ \neg S \longrightarrow A = \mathbb{R}$, $B = \mathbb{R}$
(iii) $\neg I\ \&\ S \longrightarrow A = \mathbb{R}$, $B = [-1, 1]$
(iv) $I\ \&\ S \longrightarrow A = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $B = [-1, 1]$

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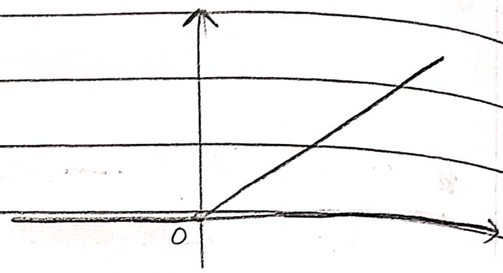
Q. ① $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x + \sqrt{x^2}$

Comment on nature of f.

A1. $f(x) = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$ $\Rightarrow$ $(\neg I)$ & $(\neg S)$



04/05/2023 ALGEBRA OF FXN^s

$$f(x) = \begin{cases} x+1, & x \geq 1 \\ -2x^2+1, & x < 1 \end{cases} \qquad g(x) = \begin{cases} 2x^3+1, & x \geq 0 \\ 2x^2-2, & x < 0 \end{cases}$$

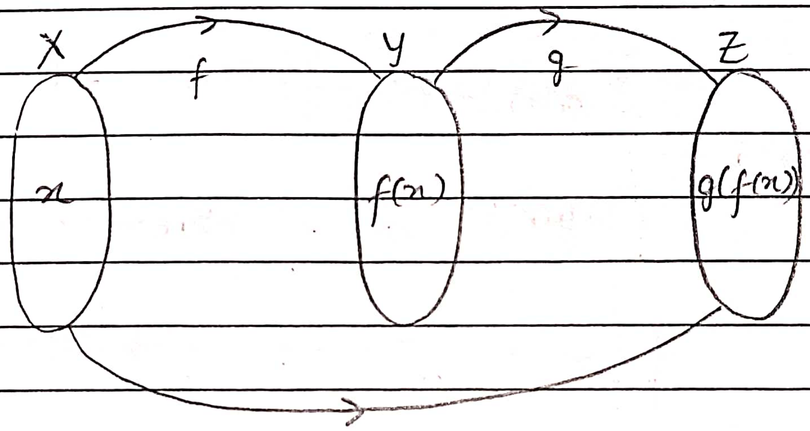
To operate on fxn^s, we redefine them as follows.

$$f(x) = \begin{cases} -2x^2+1 & x < 0 \\ -2x^2+1 & x \in [0,1) \\ x+1 & x \geq 1 \end{cases} \qquad g(x) = \begin{cases} 2x^2-2, & x < 0 \\ 2x^3+1, & x \in [0,1) \\ 2x^3+1, & x \geq 1 \end{cases}$$

NOTE: for $\frac{f(x)}{g(x)}$ we need to remove pts. from domain at which $g(x) = 0$

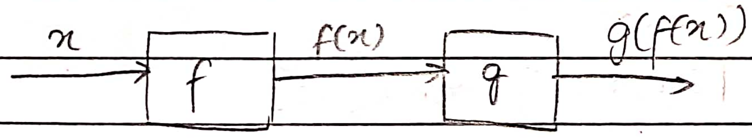
COMPOSITE FX^n

Consider 2 fx^n's $f: X \to Y$ & $g: Y \to Z$, s.t.
$h(x) = g(f(x)) = (g \circ f)(x)$



To obtain $h: X \to Z$, $h(x)$, we first take f-image of an elem $x \in X$, s.t. $f(x) \in Y$, which is the domain of $g(x)$.

Then we take g-image of $f(x)$, i.e $g(f(x))$ which would be an elem. of Z



The fx^n h defined in the diagram, called composition of f & g & is denoted by gof.
Here,

$$\text{Domain } gof(x) = \{x : x \in D(f),\ f(x) \in D(g)\}$$

Similarly, fog can be defined.

eg.

$$f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases}$$

$$g(x) = \begin{cases} x^2, & x \in [-1, 2) \\ x+2, & x \in [2, 3] \end{cases}$$

$$f(g(x)) = \begin{cases} g(x)+1 ; & g(x) \le 1 \\ 2g(x)+1 ; & g(x) \in (1, 2] \end{cases}$$

$$= \begin{cases} x^2+1 ; & x^2 \le 1, \ x \in [-1, 2) \Rightarrow x \in [-1, 1] \\ (x+2)+1 ; & x+2 \le 1, \ x \in [2, 3] \Rightarrow x \in \emptyset \\ 2x^2+1 ; & x^2 \in (1, 2], \ x \in [-1, 2) \Rightarrow x \in (1, \sqrt{2}] \\ 2(x+2)+1 ; & x+2 \in (1, 2], \ x \in [2, 3] \Rightarrow x \in \emptyset \end{cases}$$

$$= \begin{cases} x^2+1, & x \in [-1, 1] \\ 2x^2+1, & x \in (1, \sqrt{2}] \end{cases}$$

$$g(f(x)) = \begin{cases} f(x)^2 , & f(x) \in [-1,2) \\ f(x)+2 , & f(x) \in [2,3] \end{cases}$$

$$= \begin{cases} (x+1)^2 , & x+1 \in [-1,2) \Rightarrow x \in [-2,1) , \quad x \le 1 \Rightarrow \underline{x \in [-2,1)} \\ (2x+1)^2 , & 2x+1 \in [-1,2) \Rightarrow x \in [-1,\tfrac{1}{2}) , \quad x \in (1,2] \Rightarrow \underline{x \in \phi} \\ (x+1)+2 , & x+1 \in [2,3] \Rightarrow x \in [1,2] , \quad x \le 1 \Rightarrow \underline{x = 1} \\ (2x+1)+2 , & 2x+1 \in [2,3] \Rightarrow x \in [\tfrac{1}{2},1] , \quad x \in (1,2] \Rightarrow \underline{x \in \phi} \end{cases}$$

$$= \begin{cases} (x+1)^2 , & x \in [-2,1) \\ x+3 , & x = 1 \end{cases}$$

Q. If $g(x) = 1 + x - [x]$ & $f(x) = \begin{cases} -1 , & x<0 \\ 0 , & x=0 \\ 1 , & x>0 \end{cases}$

find $f(g(x))$

A. $f(g(x)) = \begin{cases} -1 , & g(x)<0 \\ 0 , & g(x)=0 \\ 1 , & g(x)>0 \end{cases}$

$g(x) = 1 - x - [x]$
$= \underline{1 + \{x\}}$

$= 1$

Q. If $f(x) = \begin{cases} 2+x, & x \geq 0 \\ 2-x, & x < 0 \end{cases}$

Find $f(f(x))$

A. $f(f(x)) = \begin{cases} 2+f(x), & f(x) \geq 0 \\ 2-f(x), & f(x) < 0 \end{cases}$

$= \begin{cases} 2+(2+x), & 2+x \geq 0, \ x \geq 0 \Rightarrow \underline{x \geq 0} \\ & \Rightarrow x \geq -2 \\ 2+(2-x), & 2-x \geq 0, \ x < 0 \Rightarrow \underline{x < 0} \\ & \Rightarrow x \leq 2 \\ 2-(2+x), & 2+x < 0, \ x \geq 0 \Rightarrow x \in \phi \\ & \Rightarrow x < -2 \\ 2-(2-x), & 2-x < 0, \ x < 0 \Rightarrow \underline{x \in \phi} \\ & \Rightarrow x > 2 \end{cases}$

$= \begin{cases} x+4 ; & x \geq 0 \\ 4-x ; & x < 0 \end{cases}$

EVEN, ODD & SYMMETRIC FX^n

Symmetric : A fx^n $y = f(x)$ is said to be symmetric about $x = a$ if

$$\boxed{f(a-x) = f(a+x)}$$

Let $f: X \to Y$, $y = f(x)$ is said to be an even/odd fnⁿ if $\exists\ (-x) \in X\ \forall\ x \in X$ s.t

Even: $f(-x) = f(x)$

Odd: $f(-x) = -f(x)$

NOTE: ① Even fnⁿ is always sym. abt. y-axis.

② Odd fnⁿ is always sym. abt origin

③ Product of 2 odd or 2 even fnⁿ. is even

④ Product of odd & even fnⁿ is odd

⑤ Every fnⁿ $y = f(x)$ can be expressed as sum of even & odd fnⁿ

$$\boxed{f(x) = \underbrace{\left[\frac{f(x)+f(-x)}{2}\right]}_{\text{Even}} + \underbrace{\left[\frac{f(x)-f(-x)}{2}\right]}_{\text{Odd}}}$$

⑥ $f(x) = 0$ is both odd & even

⑦ $f(x)$ is Even $\Leftrightarrow$ $f'(x)$ is Odd

$f(x)$ is Odd $\Leftrightarrow$ $f'(x)$ is Even

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$f(x)$	$g(x)$	$f(x)+g(x)$	$f(x)-g(x)$	$f(x)\cdot g(x)$	$\frac{f(x)}{g(x)}$	$g(f(x))$	$f(g(x))$
O	O	O	O	E	E	O	O
E	E	E	E	E	E	E	E
O	E	—	—	O	O	E	E
E	O	—	—	O	O	E	E

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PERIODIC fx^n

A fx^n $y=f(x)$ is said to be periodic if $\exists$ $T \in \mathbb{R}^+$ s.t.

$$\boxed{f(x+T)=f(x)} \quad \forall\ x \in D(f)$$

fundamental Period - smallest possible period

fx^n	Period
$s^n x$, $c^n x$, $\sec^n(x)$, $\operatorname{cosec}^n(x)$	π if $n \in$ even 2π if $n \in$ odd
$t^n x$, $\cot(x)$	π
$\lvert sx\rvert$, $\lvert cx\rvert$, ... $\lvert\cot(x)\rvert$	π
$\{x\}$	1
Constant fx^n	Periodic but w/o well defined fundamental period

- Points :-

① If $(f(x), T)$ is periodic, then

$$cf(x) \longrightarrow T$$
$$f(x+c) \longrightarrow T$$
$$f(x) \pm c \longrightarrow T$$
$$kf(ax+b) \longrightarrow T/|a|$$

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② If $(f_1(x), T_1)$ & $(f_2(x), T_2)$ are periodic,

$\Rightarrow$ $h(x) = f_1(x) \pm f_2(x)$ is periodic with $\boxed{T = LCM(T_1, T_2)}$

NOTE: (i) $LCM\left(\frac{a_1}{a_2}, \frac{b_1}{b_2}, \frac{c_1}{c_2}\right) = \frac{LCM(a_1, b_1, c_1)}{HCF(a_2, b_2, c_2)}$

(ii) $LCM(x_1, x_2)$, $x_1, x_2 \in \mathbb{Q}$ or $x_1, x_2 \notin \mathbb{Q}$ is possible

(iii) $LCM(x_1, x_2)$, $x_1 \in \mathbb{Q}$ & $x_2 \notin \mathbb{Q}$ is not possible

③ In ②, $LCM(T_1, T_2)$ is not necessarily the fundamental period.

eg. $h(x) = |sx| + |cx|$ $\qquad T_f = \pi/2$

④ Periodicity of composite fn

$$h(x) = f(g(x))$$

$(f(x), T_1)$ & $(g(x), T_2)$ $\Rightarrow$ $T = T_2$ (not necessarily T_f) $\qquad$ eg. $h(x) = c(sx)$

Only $(g(x), T_2)$ $\Rightarrow$ $T = T_2$ (not necessarily T_f) $\qquad$ eg $h(x) = |sx|$

Only $(f(x), T_1)$ $\Rightarrow$ Can be both periodic & non-periodic $\qquad$ eg $h(x) = \cos|x|$, $\sin|x|$

Neither periodic $\Rightarrow$ Can be both periodic & non-periodic $\qquad$ eg $h(x) = \overbrace{|s\underbrace{|x|}_{g(x)}|}^{f(x)}$

$$\text{periodic with 'T'}$$

☆ (5) $h(x) = f(\underbrace{g(x)}_{\text{periodic with 'T'}}) \Rightarrow$ Periodic with $\boxed{T_f = T}$

$\uparrow$ monotonic

(either increasing or decreasing but <u>not</u> both)

<u>eg</u>: $h(x) = e^{sx}$

INVERSE OF A FX^n

Let $f: A \to B$ be a one-one & onto fx^n, then $\exists$ unique fx^n $g: B \to A$ s.t.

$$f(x) = y \iff g(y) = x \quad \forall \quad x \in A \ \& \ y \in B$$

Here, 'g' is known as inverse of 'f'.

Thus, $g = f^{-1}: B \to A = \{(f(x), x) : (x, f(x)) \in f\}$

• Points :-

1. Inverse is always unique.

2. Let (h, k) be a pt. on the graph of $f(x)$, then (k, h) is the corresponding pt. on graph of $g(x)$

$$\Updownarrow$$

Symmetry of $f(x)$ & $g(x)$ about $x = y$ line

☆ (3) $f: A \to B$ is bijective & $g: B \to A$ is inverse of f

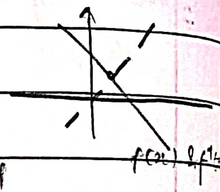
$$\Rightarrow \boxed{fog = I_B} \quad \& \quad \boxed{gof = I_A}$$

$$\Rightarrow f(f^{-1}(x)) = x \quad \& \quad f^{-1}(f(x)) = x$$

☆ (4) The graph of $y = f(x)$ & $y = f^{-1}(x)$, if they intersect, then they meet at the line $y = x$

(not necessarily only at line $y = x$)

eg. $f(x) = -x+1$ (self-inverse) $\Rightarrow$

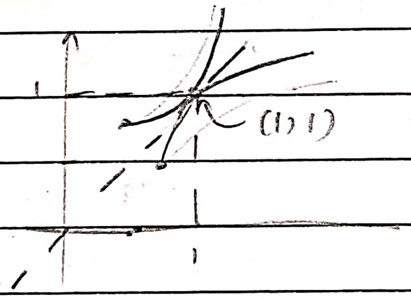


Q. $f: [\frac{1}{2}, \infty) \to [\frac{3}{4}, \infty)$, $f(x) = x^2 - x + 1$

find the inverse of $f(x)$ if it exists.

Hence or otherwise, solve $x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$

A.



$f(x) = x^2 - x + 1$

$\Rightarrow f(f^{-1}(x)) = (f^{-1}(x))^2 - f(x) + 1$

$\Rightarrow f^{-1}(x) = \frac{1}{2} \pm \sqrt{\frac{x-3}{4}}$

since, $f^{-1}: [\frac{3}{4}, \infty) \to [\frac{1}{2}, \infty)$

$\Rightarrow f^{-1}(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$

Hence, we have to solve,

$f(x) = f^{-1}(x) \quad \equiv \quad f(x) = x$

solving $\Rightarrow x^2 - x + 1 = x$

$\Rightarrow (x-1)^2 = 0$

$\Rightarrow \boxed{x = 1}$